An Algorithm for Demodulation of Correlated Quadrature Interferometer Signals

[1,2]G. Wimmer, [3]V. Witkovský, [4]R. Köning

[1]Mathematical Institute, Slovak Academy of Sciences, Bratislava, Slovakia
[2]Faculty of Natural Sciences, Matej Bel University, Banská Bystrica, Slovakia
[3]Institute of Measurement Science, Slovak Academy of Sciences, Bratislava, Slovakia
[3]Physikalisch-Technische Bundesanstalt, Braunschweig, Germany
Email: wimmer@mat.savba.sk

Abstract. *The measurement signals of the quadrature homodyne interferometers (say x and y, usually called Sine/Cosine signals and/or quadrature signals) typically exhibit offsets, unequal amplitudes and a phase difference that is not exactly* 90 *degree as would be expected in the ideal/theoretical case. Moreover, frequently there is a significant component of the measurement noise which is common to both signals (caused, e.g., by the amplitude noise of the laser), and as such, it results in a non vanishing correlation of the measured signals. Here we present a method for estimation of the unknown correlation coefficient from the observed data and suggest its implementation into the algorithm for demodulation and evaluation of the amplitude noise related uncertainty contribution of correlated quadrature interferometer signals, originally proposed by Köning, Wimmer and Witkovský in [2] for uncorrelated interferometer signals.*

Keywords: Quadrature homodyne interferometers, Heydemann correction, Ellipse fitting, Correlated interferometer signals, MINQUE

1. Introduction

In order to demodulate the observed homodyne interference signals an ellipse is fitted to both signals, x and y, simultaneously. This procedure was originally proposed by Heydeman [1] and is therefore known as Heydeman Correction (HC). The estimated ellipse parameters are used for demodulation of the quadrature interferometer signals and also for derivation of the associated uncertainties of the interferometric phase values and/or displacements (the parameters of primary interest in dimensional metrology), for more details see e.g. [1, 7]. In [2], we have suggested an iterative algorithm based on linearization of the originally nonlinear model (in fact the linear regression model with *nonlinear* constraints on its parameters). The nonlinear model is approximated locally by a linear regression model with *linear* constraints of type II, as suggested by Kubáček in [5], pp. 146 and 152. This allows to derive the *locally* best linear unbiased estimators (BLUEs) of the model (ellipse) parameters, as well as derivation of the (approximate) covariance matrix of the estimators. Using this solution the required interferometric phase values follow from (2), and their uncertainties can be obtained in a straightforward way by the law of propagation of uncertainty. The process of linearization/estimation can be iterated, until an adequately chosen convergence criterion is reached.

Originally, this method was suggested and derived for uncorrelated interferometer signals. However, as it was already mentioned, a component of the measurement noise common to both signals leads to (sometimes strongly) correlated measurement signals. So, in [3] we have modified the algorithm and presented a MATLAB implementation (`ellipseFit4HC`) which allows fitting also correlated interferometer signals, assuming that the correlation parameter is known in advance.

In situations when the number of measurements is sufficiently large, a simple analytic expression for the statistical uncertainty of the phase was derived in [4]. This allows to identify a practical limit of optical quadrature displacement interferometry, which already has been reached experimentally.

Mathematically, the (noiseless) output signals can be described as

$$\begin{aligned} x(\varphi) &= \alpha_0 + \alpha_1 \cos\varphi \\ y(\varphi) &= \beta_0 + \beta_1 \sin(\varphi + \varphi_0), \end{aligned} \tag{1}$$

where φ is the phase (the parameter of a primary interest), α_0 and β_0 denote the coordinates of the ellipse center (the offsets), α_1 and β_1 are the signal amplitudes, and $-\pi/2 < \varphi_0 < \pi/2$ is the phase offset. Under these circumstances, given the true values of the ellipse parameters, $\alpha_0, \beta_0, \alpha_1, \beta_1, \varphi_0$, and the particular signal values x and y (lying on this specific ellipse), the required interferometric phase φ is determined by using the relation

$$\varphi = \arctan\left[\frac{\alpha_1(y-\beta_0) - \beta_1(x-\alpha_0)\sin\varphi_0}{\beta_1(x-\alpha_0)\cos\varphi_0}\right]. \tag{2}$$

However, real applications have to use noisy experimental data (x_i, y_i), $i = 1, \dots, n$. So it is a problem of fitting an ellipse to data by minimizing $SS(\vartheta) = \sum_{i=1}^{n} [x_i - (\alpha_0 + \alpha_1 \cos\varphi_i)]^2 + [y_i - (\beta_0 + \beta_1 \sin(\varphi_i + \varphi_0))]^2$. The procedure requires a minimization in the $(n+5)$-dimensional parameter space, with the parameters $\vartheta = (\alpha_0, \beta_0, \alpha_1, \beta_1, \varphi_0, \varphi_1, \dots, \varphi_n)$. This is predictably cumbersome for relatively large n (a typical case for the interferometric measurements), so we shall rely on our approximation method. Here we focus mainly on the problem how to estimate the unknown correlation coefficient from the observed data.

2. Subject and Methods

We consider the following measurement model for the correlated quadrature output signals (x_i, y_i), $i = 1, \dots, n$,

$$\begin{aligned} x_i &= \mu_i + \varepsilon_{x,i}, \\ y_i &= \nu_i + \varepsilon_{y,i}, \end{aligned} \tag{3}$$

with the following set of nonlinear restrictions on the model parameters,

$$\mu_i^2 + B\nu_i^2 + C\mu_i\nu_i + D\mu_i + F\nu_i + G = 0, \quad i = 1, \dots, n, \tag{4}$$

where B, C, D, F, G represent the algebraic ellipse parameters. Notice that the ellipse parameters B, C, D, F, G only appear in the restrictions. They are uniquely related to the geometric ellipse parameters $\alpha_0, \beta_0, \alpha_1, \beta_1, \varphi_0$, for more details see [2]. In a matrix notation we get

$$\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} = \begin{pmatrix} \mu \\ \nu \end{pmatrix} + \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \end{pmatrix}, \quad \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \end{pmatrix} \sim N\left(\begin{pmatrix} \mathbf{o} \\ \mathbf{o} \end{pmatrix}, \sigma^2 \begin{pmatrix} \boldsymbol{I} & \rho\boldsymbol{I} \\ \rho\boldsymbol{I} & \boldsymbol{I} \end{pmatrix} \right) \tag{5}$$

with $\boldsymbol{x} = (x_1, \dots, x_n)'$, $\boldsymbol{y} = (y_1, \dots, y_n)'$, $\mu = (\mu_1, \dots, \mu_n)'$, $\nu = (\nu_1, \dots, \nu_n)'$, $\varepsilon_x = (\varepsilon_{x,1}, \dots, \varepsilon_{x,n})'$, $\varepsilon_y = (\varepsilon_{y,1}, \dots, \varepsilon_{y,n})'$, such that $\varepsilon_x \sim N(\mathbf{o}, \sigma^2\boldsymbol{I})$ and $\varepsilon_y \sim N(\mathbf{o}, \sigma^2\boldsymbol{I})$ (possibly correlated, with $corr(\varepsilon_{x,i}, \varepsilon_{y,i}) = \rho$, $i = 1, \dots, n$), and with nonlinear restriction on the model parameters of the form $\boldsymbol{B}\theta + \boldsymbol{b} = \mathbf{o}$, where $\boldsymbol{B} = [\nu^2 \vdots \mu\nu \vdots \mu \vdots \nu \vdots \mathbf{1}]$, $\theta = (B, C, D, F, G)'$, $\boldsymbol{b} = \mu^2$, $\mu^2 = (\mu_1^2, \dots, \mu_n^2)'$, $\nu^2 = (\nu_1^2, \dots, \nu_n^2)'$, $\mu\nu = (\mu_1\nu_1, \dots, \mu_n\nu_n)'$, and $\mathbf{1} = (1, \dots, 1)'$, $\mathbf{o} = (0, \dots, 0)'$. Here, $[\boldsymbol{u} \vdots \boldsymbol{v}]$ denotes the concatenation of the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ to a matrix.

We shall linearize the nonlinear system of restrictions, $\boldsymbol{B}\theta + \boldsymbol{b} = \mathbf{o}$, by the first-order Taylor expansion about μ_0, ν_0, and θ_0,

$$\boldsymbol{B}\theta + \boldsymbol{b} \approx \boldsymbol{A}_0 \begin{pmatrix} \mu_\Delta \\ \nu_\Delta \end{pmatrix} + \boldsymbol{B}_0 \theta_\Delta + \boldsymbol{c}_0, \tag{6}$$

where

$$\begin{aligned} \boldsymbol{A}_0 &= \left[\mathrm{Diag}\left(\left[\mathbf{o} \vdots \nu_0 \vdots \mathbf{1} \vdots \mathbf{o} \vdots \mathbf{o} \right] \theta_0 + 2\mu_0 \right) \vdots \mathrm{Diag}\left(\left[2\nu_0 \vdots \mu_0 \vdots \mathbf{o} \vdots \mathbf{1} \vdots \mathbf{o} \right] \theta_0 \right) \right], \\ \mu_\Delta &= \mu - \mu_0, \; \nu_\Delta = \nu - \nu_0, \\ \boldsymbol{B}_0 &= \left[\nu_0^2 \vdots \mu_0 \nu_0 \vdots \mu_0 \vdots \nu_0 \vdots \mathbf{1} \right], \\ \theta_\Delta &= \theta - \theta_0, \; \boldsymbol{b}_0 = \mu_0^2, \; \boldsymbol{c}_0 = \boldsymbol{B}_0 \theta_0 + \boldsymbol{b}_0 \text{ and } \theta_0 = (B_0, C_0, D_0, F_0, G_0)'. \end{aligned} \tag{7}$$

Thus, we get the (approximate) linear regression model with linear constraints,

$$\begin{pmatrix} \boldsymbol{x}_\Delta \\ \boldsymbol{y}_\Delta \end{pmatrix} \overset{approx}{\sim} N\left(\begin{pmatrix} \mu_\Delta \\ \nu_\Delta \end{pmatrix}, \boldsymbol{H} \right) \wedge \boldsymbol{A}_0 \begin{pmatrix} \mu_\Delta \\ \nu_\Delta \end{pmatrix} + \boldsymbol{B}_0 \theta_\Delta + \boldsymbol{c}_0 = \mathbf{o}, \tag{8}$$

where $\boldsymbol{x}_\Delta = \boldsymbol{x} - \mu_0$, $\boldsymbol{y}_\Delta = \boldsymbol{y} - \nu_0$, $\boldsymbol{A}_0$, $\boldsymbol{B}_0$, and $\boldsymbol{c}_0$ are given by (7), and $\boldsymbol{H}$ is the correlation matrix of the measurement errors $(\varepsilon_x', \varepsilon_y')'$, here

$$\boldsymbol{H} = \sigma^2 \boldsymbol{I}_{2n,2n} + \delta \begin{pmatrix} \mathbf{o}_{n,n} & \boldsymbol{I}_{n,n} \\ \boldsymbol{I}_{n,n} & \mathbf{o}_{n,n} \end{pmatrix} = \sigma^2 \boldsymbol{V}_1 + \delta \boldsymbol{V}_2 \tag{9}$$

with $\delta = \sigma^2 \rho$. This model serves as a first-order approximation to the nonlinear model (3)–(4). Hence, the (locally) best linear unbiased estimators (BLUEs) of the model parameters and their covariance matrix can be estimated by a method suggested in [5], for more details see also [2]:

$$\begin{pmatrix} \begin{pmatrix} \hat{\mu}_\Delta \\ \hat{\nu}_\Delta \end{pmatrix} \\ \hat{\theta}_\Delta \end{pmatrix} = -\begin{pmatrix} \boldsymbol{H}\boldsymbol{A}_0' \boldsymbol{Q}_{11,0} \\ \boldsymbol{Q}_{21,0} \end{pmatrix} \boldsymbol{c}_0 + \begin{pmatrix} \boldsymbol{I} - \boldsymbol{H}\boldsymbol{A}_0' \boldsymbol{Q}_{11,0} \boldsymbol{A}_0 \\ -\boldsymbol{Q}_{21,0} \boldsymbol{A}_0 \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_\Delta \\ \boldsymbol{y}_\Delta \end{pmatrix}, \tag{10}$$

where $\boldsymbol{Q}_{11,0}$ and $\boldsymbol{Q}_{21,0}$ are blocks of the matrix $\boldsymbol{Q}_0$ defined by

$$\boldsymbol{Q}_0 = \begin{pmatrix} \boldsymbol{Q}_{11,0} & \boldsymbol{Q}_{12,0} \\ \boldsymbol{Q}_{21,0} & \boldsymbol{Q}_{22,0} \end{pmatrix} = \begin{pmatrix} \boldsymbol{A}_0 \boldsymbol{H} \boldsymbol{A}_0' & \boldsymbol{B}_0 \\ \boldsymbol{B}_0' & \mathbf{0} \end{pmatrix}^{-1}, \tag{11}$$

together with its covariance matrix

$$Cov \begin{pmatrix} \begin{pmatrix} \hat{\mu}_\Delta \\ \hat{\nu}_\Delta \end{pmatrix} \\ \hat{\theta}_\Delta \end{pmatrix} = \begin{pmatrix} \boldsymbol{H} - \boldsymbol{H}\boldsymbol{A}_0' \boldsymbol{Q}_{11,0} \boldsymbol{A}_0 \boldsymbol{H} & -\boldsymbol{H}\boldsymbol{A}_0' \boldsymbol{Q}_{12,0} \\ -\boldsymbol{Q}_{21,0} \boldsymbol{A}_0 \boldsymbol{H} & -\boldsymbol{Q}_{22,0} \end{pmatrix}. \tag{12}$$

Then, the estimators of the original parameters μ, ν, and θ are given by $\hat{\mu} = \hat{\mu}_\Delta + \mu_0$, $\hat{\nu} = \hat{\nu}_\Delta + \nu_0$, $\hat{\theta} = \hat{\theta}_\Delta + \theta_0$.

Let σ_0^2, δ_0 be selected appropriate initial values of σ^2 and δ. Now, we shall derive the estimator (σ_0^2, δ_0)-MINQUE, i.e. the (σ_0^2, δ_0)-locally minimum norm quadratic unbiased estimator of σ^2, δ, which is optimum in the class of quadratic estimators of variance components. For more details see [6], Chapter 5.2, pp. 93-99.

For his purpose, first we shall create the 2×2 matrix $\boldsymbol{S}$, where

$$\{\boldsymbol{S}\}_{i,j} = Trace\left[\boldsymbol{A}_0'\boldsymbol{Q}_{11,0}\boldsymbol{A}_0\boldsymbol{V}_i\boldsymbol{A}_0'\boldsymbol{Q}_{11,0}\boldsymbol{A}_0\boldsymbol{V}_j\right], \quad i,j \in \{1,2\}, \tag{13}$$

and $Q_{11,0}$ is a block of matrix $\boldsymbol{Q}_0$ defined by (11) with using $\boldsymbol{H} \equiv \boldsymbol{H}_0 = \sigma_0^2\boldsymbol{V}_1 + \delta_0\boldsymbol{V}_2$. Then, the (σ_0^2, δ_0)-MINQUE of the variance components σ^2 and δ is given by

$$\begin{pmatrix}\hat{\sigma}^2\\ \hat{\delta}\end{pmatrix} = \boldsymbol{S}^{-1}\begin{pmatrix}\left(\begin{pmatrix}\boldsymbol{x}_\Delta\\ \boldsymbol{y}_\Delta\end{pmatrix} - \hat{\gamma}\right)'\boldsymbol{H}_0^{-1}\boldsymbol{V}_1\boldsymbol{H}_0^{-1}\left(\begin{pmatrix}\boldsymbol{x}_\Delta\\ \boldsymbol{y}_\Delta\end{pmatrix} - \hat{\gamma}\right)\\ \left(\begin{pmatrix}\boldsymbol{x}_\Delta\\ \boldsymbol{y}_\Delta\end{pmatrix} - \hat{\gamma}\right)'\boldsymbol{H}_0^{-1}\boldsymbol{V}_2\boldsymbol{H}_0^{-1}\left(\begin{pmatrix}\boldsymbol{x}_\Delta\\ \boldsymbol{y}_\Delta\end{pmatrix} - \hat{\gamma}\right)\end{pmatrix}, \tag{14}$$

where

$$\hat{\gamma} = \left(\boldsymbol{I}_{2n,2n} - \boldsymbol{H}_0\boldsymbol{A}_0'\boldsymbol{Q}_{11,0}\boldsymbol{A}_0\right)\begin{pmatrix}\boldsymbol{x}_\Delta\\ \boldsymbol{y}_\Delta\end{pmatrix} - \boldsymbol{H}_0\boldsymbol{A}_0'\boldsymbol{Q}_{11,0}\boldsymbol{c}_0. \tag{15}$$

The process can be iterated until convergence is reached. We should start with appropriate values $\mu_0^{(0)}$, $\nu_0^{(0)}$, $\theta_0^{(0)}$, σ_0^2, $\delta_{(0)}$. Such we obtain the (locally) BLUEs of the parameters μ, ν, θ and the (iterated) MINQUEs with their estimated covariance matrices.

3. Discussion

In this paper we have derived (and suggested to use) the explicit form of the (iterated) MINQUE estimator for estimation of the unknown correlation coefficient of the interferometer signals x and y. This helps to improve the previously suggested algorithm for demodulation and uncertainty evaluation of correlated quadrature interferometer signals.

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